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Axisymmetric critical cavities for water exclusion in "Green and Ampt" soils: use of Pologii's integral transform

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Abstract An analytical solution of Laplace's equation is obtained for the flow of water in the tension-saturated zone of a "Green and Ampt" soil, subject to uniform vertical infiltration from above, around an axisymmetric cavity of critical shape that just excludes water. The solution is obtained by converting a line-source potential in a plane seepage flow into a line source in an axisymmetric flow (the Polubarinova-Kochina solution) using Pologii's integral transform combined with a unit-gradient potential for downward seepage flow. The analysis shows that both the cavity surface and the capillary fringe boundary are paraboloids between which is sandwiched a tension-saturated region. The critical cavity obtained for the Green and Ampt soil and Philip's paraboloidal cavity obtained for a "Gardner" soil allow the estimates of the soil parameters used in the two soil models to be related.

Keywords Axisymmetric flow · Free boundary · Integral transform · Seepage

1 Introduction

Engineering excavations for subsurface structures, as well as natural cavities in soils formed by weathering, agricultural cultivations, the burrowing of worms and animals, and root development, give rise to open channels within the soil that affect unsaturated seepage flow during downward infiltration. The downward flow of water will not seep into such cavities if they are of critical shape with their walls being stream surfaces at zero soil–water pressure [1]. Generally these cavities are three-dimensional and often can be considered to be axisymmetric about a vertical axis. Philip et al. [2] investigated the critical shape for this case for a "Gardner" soil [3], that is, a soil having an exponential relationship between the hydraulic conductivity and the soil–water pressure head. In this paper we analyse the flow around such a cavity situated in a "Green and Ampt" soil [4], that is, a soil that remains saturated and of a constant conductivity, although having a negative soil–water pressure, until the soil–water pressure is reduced to the air-entry value $p_f < 0$ when the conductivity decreases to zero over a negligibly small decrease in pressure.

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The constant p_{f}, which characterizes the soil, varies from centimeters for sands to several meters for clays [5, p. 37, Table 5].

The analysis of axisymmetric flows has been developed over many decades with notable Russian advances[1] [6–11], with mathematically equivalent techniques being employed in electro–magneto-statics steady-state diffusion, heat conduction and fluid dynamics (see, for example, [12, pp. 245–256], [13, Chap. VIII, pp. 135–182], [14, 15]). For a solution to the problem discussed here, we use the integral transform [10, pp. 133–136] used by Pologii [8] to convert a line-source potential in a plane seepage flow into a line source in an axisymmetric flow.

2 Pologii's transformation

With axisymmetric flow in saturated homogenous porous materials with hydraulic conductivity K, the Darcian velocity vector for flow in cylindrical coordinates (r, z) is $\vec{V} = -K\nabla h(r, z) = \nabla\Phi(r, z)$, where $h(r, z)$ is the hydraulic head given by $h(r, z) = p(r, z) + z$; here $p(r, z)$ is the pressure head and Φ is the velocity potential given by $\Phi = -Kh$. The velocity components V_r and V_z are

$$V_r = \frac{\partial\Phi}{\partial r} = \frac{1}{r}\frac{\partial\Psi}{\partial z}, \quad V_z = \frac{\partial\Phi}{\partial z} = -\frac{1}{r}\frac{\partial\Psi}{\partial r}, \tag{1}$$

where Ψ is the Stokes stream function. Φ satisfies the Laplace equation:

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{\partial^2\Phi}{\partial z^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} = 0 \tag{2}$$

and Ψ satisfies

$$\frac{\partial^2\Psi}{\partial r^2} + \frac{\partial^2\Psi}{\partial z^2} - \frac{1}{r}\frac{\partial\Psi}{\partial r} = 0. \tag{3}$$

We note that Steward [16] introduced a vector potential based on Ψ and assembled this potential using distributed singularities for which analytical expressions for Ψ are known. Pologii [8] suggested a method of obtaining analytical solutions to axisymmetric flow problems from solutions to two-dimensional plane flows. This uses the integral transformations

$$\Phi(r, z) = \int_0^r \phi(\xi, z)\frac{\mathrm{d}\xi}{\sqrt{r^2 - \xi^2}}, \quad \Psi(r, z) = \int_0^r \psi(\xi, z)\frac{\xi\,\mathrm{d}\xi}{\sqrt{r^2 - \xi^2}} \tag{4}$$

where ϕ and ψ satisfy the Laplace equation in (x, z)-coordinates

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0, \quad \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2} = 0. \tag{5}$$

Consequently, if we know the potential and stream-function solution to a plane problem, then we can immediately get a solution to the axisymmetric problem. The main hindrance in this conversion is to get physically meaningful boundary conditions for the axisymmetric case for the boundary conditions used to obtain ϕ and ψ in the plane case. In other words, the pair (Φ, Ψ), although formally satisfying (2) and (3), may not be physically meaningful in a domain, the boundaries of which are reconstructed from the solution. This is a common problem in all "inverse" methods [17, 18] where characteristics functions are obtained from integral transforms or desired singularity distributions (see, for example, [19]). Polubarinova-Kochina [10] illustrated that careless application of the Pologii method can result in physically senseless solutions. It is to be noted that (4) is the Abel transform (see, for example, [20, pp. 631–717]).

[1] Scanned images of the papers by Kliot-Dashinskii [6], Millionshikov [7], Polubarinova-Kochina [10] and Razumova-Sretenskaya [11] are uploaded to the JEM website (e-mailed to the Editor, Prof. Strack). Electronic copies of other Russian sources are available upon request.

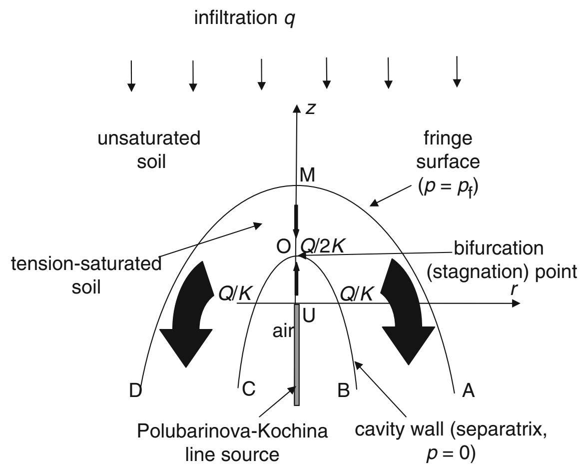

Fig. 1 Water flow around an axisymmetric critical cavity in a Green-and-Ampt soil

3 Critical shapes for cavities in downward unsaturated flows

We have found that one situation where the Pologii transform gives a solution is that for water exclusion of downward flow from axisymmetric cavities of critical shape in tension-saturated “Green and Ampt” soils when the cavities exclude water with their walls being a stream surface at zero (atmospheric) soil–water pressure. No free boundary occurs in Philip et al.’s solution [2] to the problem for a Gardner soil, unlike the solution given by the Laplace equation for a Green-and-Ampt soil that assumes a tension-saturated region bounded by a surface whose soil–water pressure equals the air-entry value for the soil.

For the two-dimensional case of exclusion from tunnel cavities Youngs and Kacimov [21] showed that the shape for a critical tunnel cavity was a parabola for a “Green and Ampt” soil, as is the case for a “Gardner” soil [2]. Additionally, Philip et al. [2] showed that the shape for an axisymmetric cavity is a paraboloid for a “Gardner” soil. We show here that this is also the case for a “Green and Ampt” soil. The relationship between the parameters in the two models for this flow situation can be obtained by matching the paraboloidal critical shapes obtained in the two results and can be compared with that deduced in other flow situations.

We consider an axisymmetric cavity, shown as BOC in Fig. 1, that sketches a vertical plane through the axis of rotation, placed in an infiltration flow of intensity q < K descending through unsaturated soil from above. Our objective is to find the critical shape of the cavity BOC on the walls of which two conditions hold, namely:

$$p = 0, \quad \Psi = 0. \tag{6}$$

In other words, Eq. 6 is a free-boundary problem which is mathematically equivalent to phreatic surface problems in groundwater hydrology.

Polubarinova-Kochina [10] considered a two-dimensional line source of intensity Q as a generating plane potential in Pologii’s transformation, placed at point $x = 0$, $z = 0$ of the (x, z)-plane, so that $\phi = Q/(2\pi) \log \sqrt{x^2 + z^2}$ where Q is the source intensity (with dimensions $\mathrm{L^2T^{-1}}$) per unit length in the direction perpendicular to the plane. She put this potential and the corresponding stream function into (4) and by direct integration arrived (omitting a constant of integration) at the following axisymmetric pair for (Φ, Ψ):

$$\Phi_{\mathrm{PK}} = \frac{Q}{2} \log\left(z + \sqrt{r^2 + z^2}\right), \quad \Psi_{\mathrm{PK}} = \frac{Q}{2}\left(z - \sqrt{r^2 + z^2}\right) \tag{7}$$

in (r, z)-coordinates to determine the flow from a line source in three dimensions, that is the flow from a semi-infinite well of an infinitesimal small radius placed at $z < 0$ in Fig. 1. Along this ray $\Phi_{\mathrm{PK}} \to -\infty$. As Polubarinova-Kochina [10] illustrated (her Fig. 3), the surfaces Φ_{PK} = constant for this flow are paraboloids with “legs” turned down and the Stokes stream surfaces Ψ_{PK} = constant are orthogonal paraboloids with “legs” turned up. In (7) Q is the intensity of the three-dimensional line source.

In order to obtain the critical cavity shape we use the fact that both (3) and (4) are linear. Thus we add to Polubarinova-Kochina's solution, (7), the potential and Stokes function of a unidirectional descending flow $\Phi_{\mathrm{u}} = -Kz$, $\Psi_{\mathrm{u}} = Kr^2/2$ to give the combined potential and Stokes function as:

$$\Phi_{\mathrm{c}} = \frac{Q}{2}\log\left(z + \sqrt{r^2+z^2}\right) - Kz, \quad \Psi_{\mathrm{c}} = \frac{Q}{2}\left(z - \sqrt{r^2+z^2}\right) + Kr^2/2. \tag{8}$$

Elementary inspection shows that (1–3) are satisfied by (8).

Now we determine the shape of the stream surface $\Psi_{\mathrm{c}} = 0$. From the second equation in (8) this surface consists of two parts:

$$\begin{aligned} &z = \frac{Q}{2K} - \frac{Kr^2}{2Q}, \quad \text{at } z < Q/2K, \\ &r = 0 \quad \text{at } z > Q/2k. \end{aligned} \tag{9}$$

The first equation in (9) describes a paraboloid, an axial section of which is shown as the separatrix in Fig. 1. The tip O of the paraboloid is located at the point $r = 0$, $z = Q/2K$. The second equation in (9) is just a straight line. The cavity nose O is a bifurcation point at which the descending unidirectional flow from above meets the ascending flow from the source.

The isobars for the flow determined by (8) are

$$p_{\mathrm{c}} = -\frac{Q}{2K}\log\left(z + \sqrt{r^2+z^2}\right) - C. \tag{10}$$

The constant C in (10) is found from the condition that at the nose O ($z = Q/2K$ in Fig. 1) the pressure head is zero giving $C = -Q/2K \log Q/K$ that upon substitution in (10) yields

$$p_{\mathrm{c}} = -\frac{Q}{2K}\log\frac{K\left(z+\sqrt{r^2+z^2}\right)}{Q}. \tag{11}$$

We now have to prove that our no-flow paraboloid in (9) is also a surface at atmospheric pressure. From (11), for $p_{\mathrm{c}} = 0$, $\frac{K(z+\sqrt{r^2+z^2})}{Q} = 1$, which is that already obtained in (9) for our Stokes stream surface. Thus (9), which is a combination of a three-dimensional dipole at infinity and a three-dimensional semi-infinite line source, gives a critical shape for the cavity on which the pore water pressure is atmospheric with flow over its surface.

We now need to prove that the seepage flow past this critical shape originates from a physically meaningful source. For this purpose, we shall find the whole family of isobars p_{c}. From (11) these isobars are given by the equation

$$z + \sqrt{r^2+z^2} = Q/K \exp[-2Kp_{\mathrm{c}}/Q]$$

or

$$z = \frac{Q}{2K}\exp[-2Kp_{\mathrm{c}}/Q] - \frac{Kr^2}{2Q}\exp[2Kp_{\mathrm{c}}/Q]. \tag{12}$$

Equation 12 describes a family of paraboloids of which our critical cavity is a limiting case ($p_{\mathrm{c}} = 0$). One contour of this family, DMA, is shown in Fig. 1. The Stokes-function distribution along surfaces (12) can be found by putting (12) into the second equation of (9) to obtain

$$\Psi_{\mathrm{u}} = \frac{Kr^2}{2}\left(1 - \exp[2Kp_{\mathrm{c}}/Q]\right). \tag{13}$$

Equation 13 describes the inflow through our isobars. Since the term in the brackets in (13) is constant, Eq. 13 is just a rescaled expression for the Stokes function of the uniform flow Ψ_{u}. In other words, water seeps uniformly through the capillary fringe surface DMA in Fig. 1 in the same manner as in the two-dimensional situation investigated in [21–23]. Thus, our paraboloids (12) are the capillary fringe surfaces where $p_{\mathrm{c}} = p_{\mathrm{f}} < 0$ with uniform accretion of intensity $(1 - \exp[2Kp_{\mathrm{f}}/Q])$. This value corresponds to the infiltration rate q/K through the unsaturated region above. Between AMD and BOC in Fig. 1 in a Green-and-Ampt soil we have a tension-saturated zone sandwiched between two paraboloids when water does not break through into the cavity, similar to the two-dimensional situation [23] of a saturated zone saturated between two parabolas.

To obtain the critical cavity for a uniform infiltration of intensity q for a soil with a given values of K and p_{f}, Q is obtained by solving

$$1-\exp[2Kp_{\mathrm{f}}/Q]=\frac{q}{K}, \tag{14}$$

so that

$$Q=\frac{2Kp_{\mathrm{f}}}{\log(1-q/K)}. \tag{15}$$

Then the cavity contour where $p_{\mathrm{c}}=0$ can be obtained from (9) as

$$z=\frac{p_{\mathrm{f}}}{\log(1-q/K)}-\frac{\log(1-q/K)}{4p_{\mathrm{f}}}r^2 \tag{16}$$

and the fringe boundary where $p_{\mathrm{c}}=p_{\mathrm{f}}$ as

$$z=\frac{p_{\mathrm{f}}}{(1-q/K)\log(1-q/K)}-\frac{(1-q/K)\log(1-q/K)}{4p_{\mathrm{f}}}r^2, \tag{17}$$

with the height z_{OM} given by

$$z_{\mathrm{OM}}=\frac{p_{\mathrm{f}}}{(1-q/K)\log(1-q/K)} \tag{18}$$

Figure 2 illustrates the flow net for $Q=1$, $p_{\mathrm{f}}=-1$, that is for $q/K=0.865$ with equipotentials and stream surfaces constructed using (8). Equipotentials and the stream surfaces are shown for values indicated by the curves. The cavity contour and capillary fringe are shown by solid lines. We note that the equipotentials at $\Phi<\Phi c$ ($\Phi_{\mathrm{c}}=-0.605$ for the example in Fig. 2) are double-connected. The flow net outside the sheath in the cavity interior and above the capillary fringe are discarded as not being part of the solution to the physical problem in the tension-saturated zone. We note that above the fringe illustrated in Fig. 1 the flow takes place in unsaturated soil as discussed for the similar plane problem [22]. The half-contours of the critical cavity and the fringe boundary between which is sandwiched the tension-saturated soil are shown in Fig. 3 for three values of the accretion rate, namely for $q/K=0.2$, 0.4 and 0.6, shown plotted in scaled coordinates $z/|p_{\mathrm{f}}|$ and $r/|p_{\mathrm{f}}|$.

For cavities in a "Gardner" soil for which Philip et al. [2] also found parabaloidal critical shapes, there is no sheath of tension-saturated soil with the soil desaturating as the soil–water pressure becomes more negative. In this case the hydraulic conductivity of the unsaturated soil K_{u} is described by $K_{\mathrm{u}}=K\exp(\alpha p)$ where α is a constant for the given soil. The paraboloidal cavity is given [2] by

$$z=-\frac{\ell}{2\alpha}+\frac{\alpha r^2}{2\ell}, \tag{19}$$

where ℓ is the dimensionless apical radius of curvature expressed as a fraction of α^{-1} given by

$$\frac{q}{K}=1-\frac{\ell}{2}\exp(\ell/2)\mathrm{E}(\ell/2), \tag{20}$$

where E is the exponential integral. The expression for the critical cavity, (16), for a cavity in a Green-and-Ampt soil coincides with that, (19), for a Gardner soil if

$$\alpha\,|p_{\mathrm{f}}|=-\frac{\ell\log(1-q/K)}{2}. \tag{21}$$

Equation 21 gives the relationship between the parameters of the two soil models in order for the computed parabaloidal cavity shapes to coincide; $\alpha|p_{\mathrm{f}}|$ is plotted against q/K in Fig. 4. The relationship given by the parabolic relationship for the two-dimensional tunnel cavity [23] and also the relationship

$$\alpha|p_{\mathrm{f}}|=1-q/K \tag{22}$$

that is obtained by comparing the macroscopic capillary lengths [24] for the two soil models are also shown.

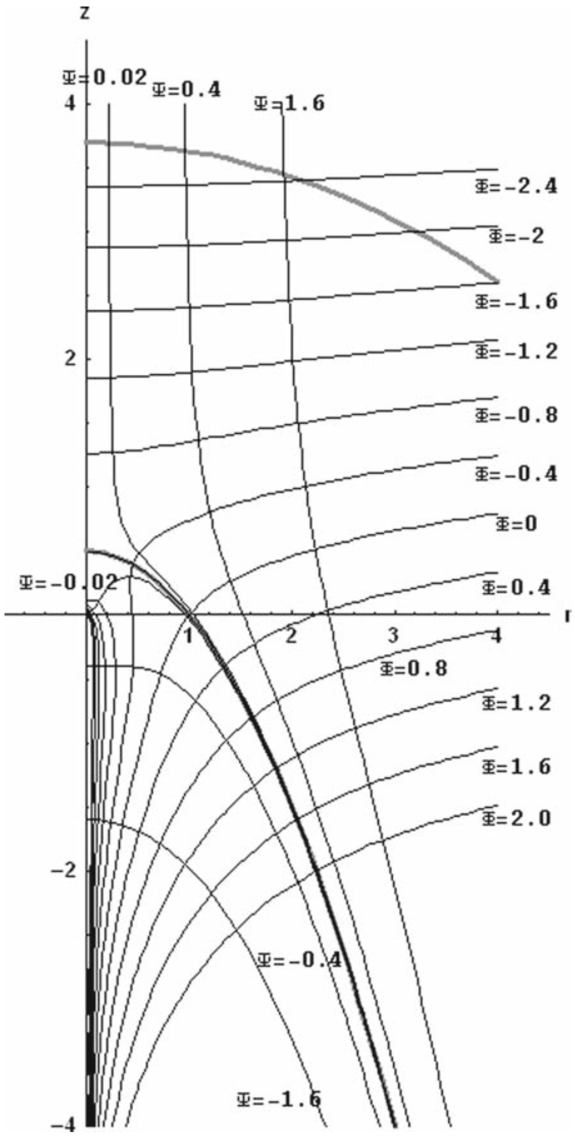

Fig. 2 Seepage flow net around a critical axisymmetric cavity for $Q = 1$, $p_\mathrm{f} = -1$ ($q/K = 0.865$)

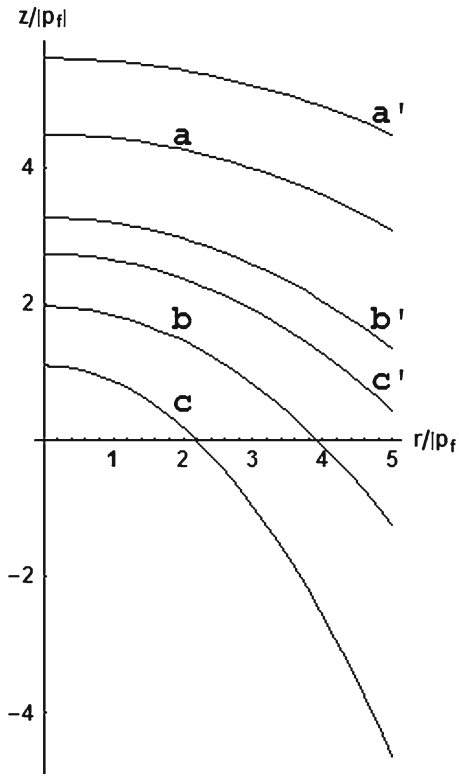

Fig. 3 The critical cavity for $q/K = 0.2$ (curve **a**), 0.4 (curve **b**) and 0.6 (curve **c**) given by Eq. 16 and the corresponding fringe surfaces (curves **a**′, **b**′ and **c**′) given by Eq. 17

4 Discussion

The use of Polubarinova-Kochina's solution [10] using Pologii's integral transform [8] to convert a line-source potential in a plane seepage flow into an axisymmetric semi-infinite line source and superimposing a unidirectional downward flow allows a solution of the Laplace equation for the velocity potential in an axisymmetric flow occurring in a sheath-shaped domain bounded by two free surfaces. One free surface, the paraboloid BOC in Fig. 1, is a critical cavity shape that is watertight and isobaric at the same time. The second free boundary of the problem, the paraboloid AMD in Fig. 1, satisfies the conditions of isobaricity and uniform accretion and represents a capillary fringe receiving water from uniform infiltration from above into a region of tension-saturated soil sheathed around

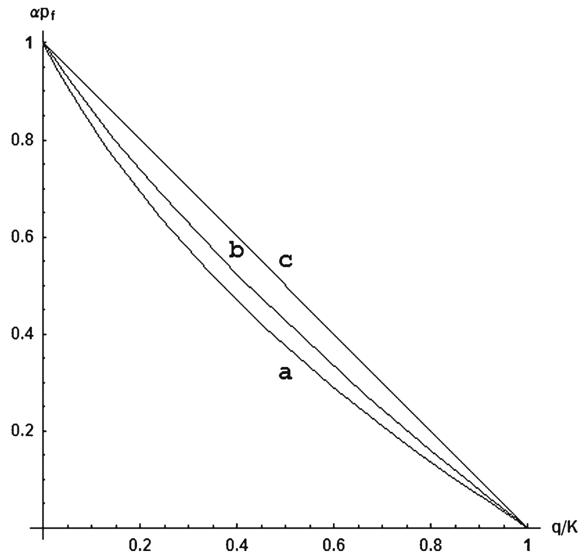

Fig. 4 The relationship between $\alpha|p_f|$ and q/K given by Eq. 21 (curve **a**) compared with that found for a two-dimensional tunnel cavity (curve **b**) and that given by Eq. 22 obtained from the macroscopic capillary lengths (curve **c**)

the cavity. It is noteworthy that the critical shapes of the cavities were found to be parabaloids for a Gardner soil [2] and that parabaloids were also obtained by Ivantsov [25, Eq. 14] for transient problems.

The Pologii transform, which has been used [10] for converting a plane line source into an axisymmetric source, can be used in other problems of groundwater hydrology, for example, the situation of a cavity under flooded conditions. However, the use of the transform with the Polubarinova-Kochina's solution [10] has resulted in addressing problems with little physical significance, such as those concerned with semi-infinite boundaries. In the case of critical cavities that this paper addresses, this is the case. However, as argued by Philip et al. [1,2], the flow regime near the nose of the paraboloid will not be so different if the cavity is truncated at some depth and his solution of the exclusion problem gives results of practical interest. This is also the case with the solution of the exclusion problem for a cavity resting on a very permeable freely draining substratum [22] that gives results that become the same near the nose as the depth to the substratum becomes great. However, the general situation of finite boundaries in groundwater hydrology requires the search for Pologii's generating potential and stream functions that transform into exact meaningful boundary conditions.

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